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DOUBLE POINTS AND LINES.

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THIS note is in continuation of the Author's Notes on the same subject published on pp. 19—21, Vol. IV and pp. 264—8, Vol. X of the Journal of the Indian Mathematical Society, and falls into three parts. The first part deals with orthologic triangles in perspective, and contains a simple construction for finding the double point of such triangles. Analytical formulæ for the double point and lines are given in the second part and applied to obtain the following interesting result :

Any two triangles ABC *and* A′B′C′ *and the triangle formed by the radical axes of pairs of circles described on* BC′, B′C ; CA′, C′A ; AB′, A′B, *have a common double point.*

In the case of two orthologic triangles a simple reciprocal relation is shown to exist between their double point and the point of concurrence of the perpendiculars from the vertices of one on the sides of the other. This relation is looked at from the view of correspondence in the last part wherein the present theory is made use of to discuss geometrically several properties of the invariant triangles occurring in the transformation of a given point into the double point of a given triangle and the pedal triangle of the point.

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PART I.

Orthologic Triangles in Perspective.

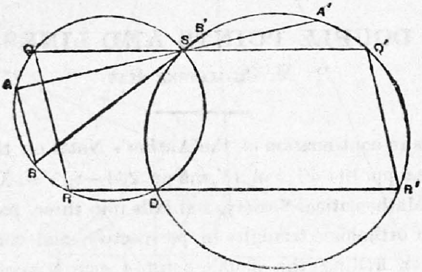
§ 1. *Construction for finding the Double Point.*

Let ABC and A'B'C' be similar triangles in perspective but not homothetic, and S the centre of perspective. It follows by Elementary Geometry [1] that

(1) S lies on the circumcircles of ABC and A'B'C' intersecting again at the double point D of the triangles.

(2) Any line through S cuts the circles in corresponding points Q and Q'.

(3) If QR and Q'R' be parallel chords of the respective circles, RR' passes through the double point D, and RA, RB, RC are respectively parallel to R'A', R'B', R'C'.



It is known [2] that triangles in perspective are conjugate with respect to a conic whose centre is the double point of the triangles.

As an immediate consequence we deduce the following theorem which results by projection :—

THEOREM I.

If two triangles ABC, A'B'C' are conjugate with respect to a conic, the diameter of the conic through A passes through a point X such that XB', XC' are respectively parallel to AB, AC.

It has also been shown [3] that the double point of orthologic triangles ABC and A'B'C' is the point of concurrence of the radical axes of pairs of circles described on BC', B'C ; CA', C'A ; AB', A'B, as diameters.

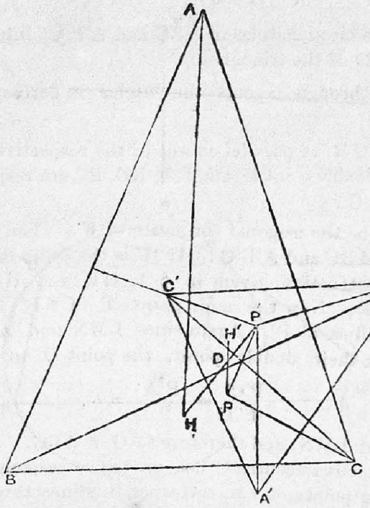
1 Cf. Lachlan : *Modern Geometry*, p. 130.

2 Cf. Author's paper "Double Lines"—Journal of the Indian Math. Soc., Vol. X, pp. 264—8.

3 *Loc. cit.* : *Jour. Ind. Math. Soc.*

If the triangles are also in perspective, the following is a simple construction for finding the double point.

Let H, H′ be the orthocentres of ABC and A′B′C′, and P,P′, the points of concurrence of the perpendiculars from A,B,C on the corresponding sides of A′B′C′, and of the perpendiculars from A′,B′,C′ on the corresponding sides of ABC. Then HP′ and H′P intersect at the double point.



Proof.—It has been shown[4] that P,P′ are corresponding points of ABC and A′B′C′, and that PP′ passes through the centre of perspective. Since AH is parallel to A′P′, it follows from § 1. (3), that HP′ passes through the double point. Similarly H′P passes through the double point.

§ 2. *Some Theorems.*

We shall now proceed to obtain a condition for two orthologic triangles defined in a particular manner to be in perspective. Thus,

THEOREM II.

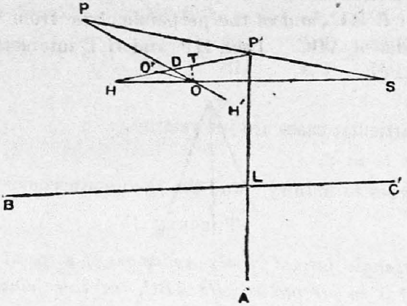
If LMN is the pedal triangle of P′ with respect to ABC, and P′L P′M, P′N be produced to A′B′C′ such that

$$\frac{P'A'}{P'L} = \frac{P'B'}{P'M} = \frac{P'C'}{P'N} = \lambda$$

[4] *Loc, cit,* p. 267—7 (3).

and the join of P′ to its isogonal conjugate meets the Euler's line (OH) in S, then if ABC and A′B′C′ are in perspective

$$\lambda = 2\,\frac{OS}{HS}.$$



Proof.—Let P be the isogonal conjugate of P′. Then P,P′ are corresponding points of ABC and A′B′C′. If H′ is the ortho-centre of A′B′C,′ then from the construction given in § 1, HP′ and H′P intersect in the double point D. Now the mid. point T of HP′ [5] is the centre of the orthopolar ellipse of P′. Again since LMN and A′B′C′ are homothetic having P′ for their double point, the point O′ on P′T defined by

$$\frac{P'O'}{P'T} = \frac{P'A'}{P'L} = \lambda,$$

corresponds to T of LMN, and therefore to O of ABC. Now O′ lies on HP′; therefore the circumcentre O lies on the corresponding line H′P. Since corresponding points on two corresponding lines through the double point are on parallel rays, OO′ is parallel to PP′.

$$\text{Hence } \frac{O'P'}{HP'} = \frac{OS}{HS}.$$

$$\text{But } \quad HP' = 2TP'.$$

$$\therefore \quad \frac{O'P'}{2TP'} = \frac{OS}{HS}.$$

$$\text{Also } \frac{O'P'}{TP'} = \frac{P'A'}{P'L} = \lambda.$$

$$\therefore \quad \lambda = 2\frac{OS}{HS}.$$

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[5] The orthopolar ellipse of P′ with respect to ABC, considered as a figure of LMN, corresponds to the circumcircle of ABC. Cf. *Jour. Ind. Math. Soc.*, Vol. IV., p. 110.

The converse of this theorem is also true, *viz.*

THEOREM III.

If the line joining P to its isogonal conjugate with respect to ABC intersects the Euler's line OH in S, and if L, M, N, the feet of the perpendiculars from P on the sides of ABC be produced to A', B', C' such that $\frac{PA'}{PL} = \frac{PB'}{PM} = \frac{PC'}{PN} = 2\frac{OS}{HS} = \lambda$, *the triangle A'B'C' is in perspective with ABC.*

Some particular cases are interesting.

(i) Let $\lambda = 2$,

then S goes to infinity ; and the theorem becomes

THEOREM IV.

If the triangle formed by the reflexions of a point P in the sides of a triangle ABC is in perspective with ABC, the line joining P to its isogonal conjugate is parallel to the Euler's Line.

(ii) Let $\lambda = 1$.

In this case HS = 2OS and S is the orthocentre of the antimedial triangle of ABC. The theorem consequently becomes

THEOREM V.

If the pedal triangle of a point P with respect to a triangle ABC is in perspective with ABC, the line joining P to its isogonal conjugate passes through the orthocentre of the antimedial triangle of ABC.

(iii) Let $\lambda = -1$.

S is here the centroid and the theorem[*] as follows :—

THEOREM VI.

If P is the centre of a conic touching the sides of a triangle ABC at points where the normals are concurrent, the triangle formed by the reflexions of the feet of the perpendiculars from P on the sides of ABC, in P, is in perspective with ABC.

PART II.

§ 1. *Introductory Analytical Formulæ.*

Let ABC and A'B'C' be two triangles. Referred to the triangle ABC, let the normal co-ordinates of A', B', C' be (l_1, m_1, n_1), (l_2, m_2, n_2),

[*] When the line joining a point to its isogonal conjugate passes through the centroid, that point is the centre of a conic touching the sides of a triangle at points where the normals are concurrent. Cf. *Jour. of Ind. Math. Soc.*, Q. 480, Vol. IV.

(l_3, m_3, n_3). If D be the point (α, β, γ), the point corresponding to D for A'B'C' is

$$\left(\frac{a\alpha l_1}{al_1 + bm_1 + cn_1} + \frac{b\beta l_2}{al_2 + bm_2 + cn_2} + \frac{c\gamma l_3}{al_3 + bm_3 + cn_3}\right.;$$

with similar expressions for the other two co-ordinates).

On writing a' for $\dfrac{a}{al_1 + bm_1 + cn_1}$,

b' for $\dfrac{b}{al_2 + bm_2 + cn_2}$,

c' for $\dfrac{c}{al_3 + bm_3 + cn_3}$.

the A'B'C' — correspondent of D is

$$(a'l_1\alpha + b'l_2\beta + c'l_3\gamma, \ldots \ldots)$$

Now if D is to be the double point, it must correspond to itself, that is to say, we should have

$$a'l_1\alpha + b'l_2\beta + c'l_3\gamma = \alpha,$$
$$a'm_1\alpha + b'm_2\beta + c'm_3\gamma = \beta,$$
$$a'n_1\alpha + b'n_2\beta + c'n_3\gamma = \gamma.$$

These equations of which only two are independent represent three lines through the double point. On solving, we find the *normal co-ordinates of the double point* to be

$$(al_1 + bm_1 + cn_1)(al_2l_3 + bl_2m_3 + cl_3n_2),$$
$$(al_2 + bm_2 + cn_2)(bm_3m_1 + cm_3n_1 + am_1l_3),$$
$$(al_3 + bm_3 + cn_3)(cn_1n_2 + an_1l_2 + bn_2m_1).$$

Similarly to find *the equation of the double lines*:—

Let $L \equiv p\alpha + q\beta + r\gamma = 0$ be a double line. If (XL) denotes the length of the perpendicular from X on L, we have

$$\frac{(A'L)}{(AL)} = \frac{(B'L)}{(BL)} = \frac{(C'L)}{(CL)}.$$

Consequently,

$$\frac{a}{p} \cdot \frac{pl_1 + qm_1 + rn_1}{al_1 + bm_1 + cn_1} = \frac{b}{q} \cdot \frac{pl_2 + qm_2 + rn_2}{al_2 + bm_2 + cn_2}$$
$$= \frac{c}{r} \cdot \frac{pl_3 + qm_3 + rn_3}{al_3 + bm_3 + cn_3} = \lambda \text{ say.}$$

These equations may be written

$$a'(pl_1 + qm_1 + rn_1) = p\lambda \quad \dots \quad \dots \quad \dots \quad (1)$$

$$b'(pl_2 + qm_2 + rn_2) = q\lambda \quad \dots \quad \dots \quad \dots \quad (2)$$

$$c'(pl_3 + qm_3 + rn_3) = r\lambda \quad \dots \quad \dots \quad \dots \quad (3)$$

together with the equation of L,

$$p\alpha + q\beta + r\gamma = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

The elimination of p, q, r, λ from these equations gives the equation of the double lines.

The elimination is most simply effected as follows :—

Multiplying (1), (2), (3) by α, β, γ respectively and adding, we have

$$p\,(a'l_1\alpha + b'l_2\beta + c'l_3\gamma) + q\,(a'm_1\alpha + b'm_2\beta + c'n_3\gamma) + r\,(a'n_1\alpha + b'n_2\beta + c'n_3\gamma) = 0 \quad \dots \quad (5)$$

If (f, g, h) is the double point, we have

$$pf + qg + rh = 0, \quad \dots \quad (6)$$

because the double line passes through the double point.

Hence eliminating p, q, r from (4), (5), (6), the equation of the double lines is

$$\begin{vmatrix} a'l_1\alpha + \dots & a'm_1\alpha + \dots & a'n_1\alpha + \dots \\ \alpha & \beta & \gamma \\ f & g & h \end{vmatrix} = 0.$$

or on expansion,

$$\alpha^2 a'(n_1 g - m_1 h) + \beta^2 b'(l_2 h - n_2 f) + \gamma^2 c'(m_3 f - l_3 g) + \beta\gamma\,(c'l_3 h - b'l_2 g + b'm_2 f - c'n_3 f) + \dots\dots\dots = 0.$$

§ 2. *Co-ordinates of the Double Point (continued.)*

Now, the double point of two triangles is usually defined as the mean centre of the vertices of those triangles for a common system of multiples. The analytical expression for those multiples is obtained from the normal co-ordinates given in § 1. The geometrical intrepretation in terms of the powers with respect to circles described on some associated lines as diameters is interesting, and a convenient proof of it is obtained from the foregoing formulæ.

THEOREM VII.

The double point D of two triangles ABC, A'B'C', is the mean centre f the vertices of either triangle for the multiples D (BC') — D (B'C), D (CA')—D (C'A), D (AB')—D (A'B) *where* X (YZ) *denotes the power of* X *with respect to the circle described on* YZ *as diameter.*

Now if

$$S \equiv (BC'^2 + CA'^2 + AB'^2) - (B'C^2 + C'A^2 + A'B^2)$$

the vanishing of S is the well-known condition for the triangles to be orthologic.

But,

$$S = 2\Delta .\ 4\ R \sum \frac{m_1 \cos B - n_1 \cos C}{al_1 + bm_1 + cn_1},$$

where Δ is the area of ABC. (1)

$$\therefore \quad S = 4\ R \Sigma\ (m_1 \cos B - n_1 \cos C)$$

$$= 4\ R \Sigma \cos A\ (l_2 - l_3).$$

employing absolute co-ordinates.

Again, it is well-known, that the triangle formed by the perpendiculars from A', B', C' on the sides of ABC is similar to ABC, the ratio of similitude, ρ, being equal to $\frac{S}{4\Delta}$. (2)

It is enough, for establishing the theorem, to show that

$$D\ (AB') - D\ (A'B) = -\ \rho ch.$$

where h is the distance of D from AB.

Now the equation of the circle (AB') is

$$\left\{ \frac{\beta}{b}\ (l_2 + n_2 \cos B) + \frac{\gamma}{c}(l_2 + m_2 \cos C) \right\}\ (a\alpha + b\beta + c\gamma)$$

$$= \frac{1}{bc}\ (al_2 + bm_2 + cn_2)\ (a\beta\gamma + b\gamma\alpha + c\alpha\beta);$$

and that of (A'B) is

$$\left\{ \frac{\gamma}{c}\ (m_1 + l_1 \cos C) + \frac{\alpha}{a}\ (m_1 + n_1 \cos A) \right\} (a\alpha + b\beta + c\gamma)$$

$$= \frac{1}{ca}\ (al_1 + bm_1 + cn_1)(a\beta\gamma + b\gamma\alpha + c\alpha\beta).$$

Consequently

$$D(AB') - D(A'B)$$

$$= \frac{abc}{af+bg+ch}\left\{\frac{\frac{g}{b}(l_2+n_2\cos B)+\frac{h}{c}(l_2+m_2\cos C)}{al_2+bm_2+cn_2}bc - \frac{\frac{h}{c}(m_1+l_1\cos C)+\frac{f}{a}(m_1+n_1\cos A)}{al_1+bm_1+cn_1}.ca\right\}$$

where f, g, h are, as before, the normal co-ordinates of D.

Substitute for f and g in the expression within the brackets their values given on p. 126 we find that

$$(l_2+n_2\cos B)(bm_3m_1+cm_3n_1+am_1l_3)-$$
$$(m_1+n_1\cos A)(al_3l_2+bl_3m_2+cl_3n_2)$$
$$=(m_3\cos B-l_3\cos A)(cn_1n_2+an_1l_2+bn_2m_1)$$
$$=\frac{(m_3\cos B-l_3\cos A)h}{al_3+bm_3+cn_3}.$$

On taking h outside the bracket, we get

$$\frac{D(AB')-D(A'B)}{h}=\frac{abc}{af+bg+ch}\left\{\frac{b(l_2+m_2\cos C)}{\Sigma al_2}-\frac{a(m_1+l_1\cos C)}{\Sigma al_1}-\frac{c(l_3\cos A-m_3\cos B)}{\Sigma al_3}\right\}.$$

The first two terms within the brackets will be found to have c as a factor. Taking this outside, we have

$$\frac{D(AB')-D(A'B)}{ch}=\frac{-abc}{af+bg+ch}\left\{\sum\frac{m_1\cos B-n_1\cos C}{al_1+bm_1+cn_1}\right\}$$

$$=\frac{-abc}{af+bg+ch}\cdot\frac{S}{8\triangle R}, \text{ by (1)}$$

$$=\frac{-S}{4\triangle}, \text{ if } f, g, h \text{ denote actual lengths.}$$

$$= -\rho, \text{ the ratio of similitude referred to in (2).}$$

Consequently $\quad D(AB') - D(A'B) = -\rho ch$

and the theorem is established.

§ 3. *Three triangles having a common double point.*

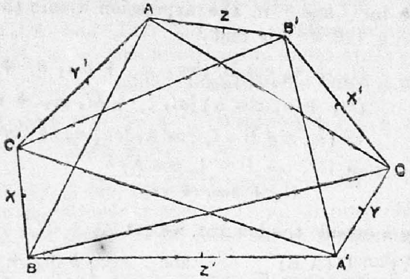
It has already been shown[7] that in the case of two orthologic triangles ABC and A′B′C′, the radical axes of the pairs of circles described on BC′, B′C, etc. as diameters, intersect in the double point of the triangles. Now

[7] *Jour. Ind. Math. Soc. loc cit.* Vol. X,

the elementary theorem in co-axal circles that the difference of the powers of a point with respect to two circles is twice the rectangle contained by its distance from their radical axis and the distance between their centres, serves to give a generalisation of this Theorem.

THEOREM VIII

Any two triangles ABC, A'B'C' *and the triangle formed by the radical axes of the pairs of circles described on* BC', B'C ; CA', C'A ; AB', A'B, *as diameters, have a common double point.*



Proof. Let X,X' Y,Y' Z,Z' be the middle points of BC', B'C etc. The triangles XYZ and X'Y'Z' have the same median point which is the mean centre of A,B,C, A',B',C'. Therefore if λ, μ, v are the angles of the triangle formed by XX', YY', ZZ', we have

$$\frac{XX'}{\sin \lambda} = \frac{YY'}{\sin \mu} = \frac{ZZ'}{\sin v}. \qquad (1)$$

The radical axes being perpendicular to XX', YY', ZZ' form a triangle, say LMN, with angles λ, μ, v. If P be any point, and l, m, n be the distances of P from the sides of LMN, P is the mean centre of LMN for the multiples

$$l \sin \lambda,\ m \sin \mu,\ n \sin v\ ;$$

i. e., $\qquad l.\,XX',\ \ m.\,YY',\ \ n.\,ZZ'$, by (1).

But

$$2\,l.\,XX' = P\,(BC') - P\,(B'C) \text{ by co-axal theory.}$$

Therefore P is the mean centre of LMN for multiples P (BC')—P (B'C), etc.

Therefore D, the double point of ABC and A'B'C' is the mean centre of LMN for multiples D (BC') — D (B'C), etc., and by Theorems VII, D is also the mean centre of ABC and A'B'C' for the same multiples. Hence D is the common double point of ABC, A'B'C' and LMN.

§ 4. *A Reciprocal Relationship..*

Let ABC and A′B′C′ be two orthologic triangles and the perpendiculars from A′, B′, C′ on BC, CA, AB meet them in L, M, N and concur at P. If P be the point (l, m, n) the normal co-ordinates of A′, B′, C′ are

$$l(1-\lambda),\ m+\lambda l\cos C,\ n+\lambda l\cos B\ ;$$
$$l+\mu m\cos C,\ m(1-\mu),\ n+\mu m\cos A\ ;$$
$$l+v n\cos B,\ m+v n\cos A,\ n(1-v)\ ;$$

where $\dfrac{PA'}{PL}=\lambda,\ \dfrac{PB'}{PM}=\mu,\ \dfrac{PC'}{PN}=v.$

Now the double point (α, β, γ) of ABC and A′B′C′ is found by substituting the above values of the co-ordinates of A′, B′, C′ in two of the equations in § 1 and solving them. Thus,

$$al(1-\lambda)\alpha+b(l+\mu m\cos C)\beta+c(l+v\,n\cos B)\gamma = \alpha(al+bm+cn), \qquad (1)$$

and a similar equation.

If the double point (α, β, γ) is given as also the quantities $\lambda,\ \mu,\ v$ P is uniquely determined by the following equations for $l,\ m,\ n$ got by rearranging the above

Thus,

$$a\alpha(1+\lambda)l+bm(\alpha-\mu\beta\cos C)+cn(\alpha-v\gamma\cos B) = (a\alpha+b\beta+c\gamma)\,l \qquad (2)$$

and a similar equation.

If we examine the equations (1) and (2), we find that (2) can be derived from (1) by merely substituting

$$\left.\begin{matrix}\alpha & \beta & \gamma\\ l & m & n\\ \lambda & \mu & v\end{matrix}\right\}\quad\text{for}\quad\left.\begin{matrix}l & m & n\\ \alpha & \beta & \gamma\\ -\lambda & -\mu & -v\end{matrix}\right\}.$$

Therefore if we are given the double point D of two orthologic triangles ABC and A′B′C′ and the ratios $\dfrac{PA'}{PL},\ \dfrac{PB'}{PM},\ \dfrac{PC'}{PN}$, P is determined by the following construction :—

From D drop DL′, DM′, DN′ perpendicular to the sides of ABC and find points A″, B″, C″ on them such that

$$\frac{DA''}{DL'}=-\frac{PA'}{PL},\ \frac{DB''}{DM'}=-\frac{PB'}{PM},\ \frac{DC''}{DN'}=-\frac{PC'}{PN}.$$

Then P is the double point of the triangles ABC and A″B″C″.

Designating the ratios $\frac{PA'}{PL}$, $\frac{PB'}{PM'}$, $\frac{PC'}{PN}$ by λ, μ, v as before, when P, λ, μ, v are given, the triangle A′B′C′ is completely determined and may be demoted by (P, λ, μ, v). If the statement that D is the double point of ABC and A′B′C′ be denoted by the equation

$$(P, \lambda, \mu, v,) = D,$$

the above results show that

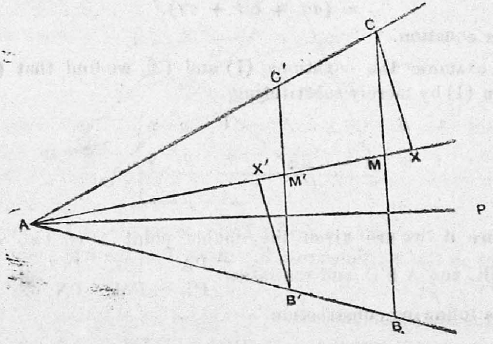
$$(D, -\lambda, -\mu, -v,) = P$$

that is to say, P and D are interchangeable with a change in the sign of λ, μ, v.

§ 5 *Geometrical Proof.*

While examining some special cases, the reciprocal relationship between P and D, of two orthologic triangles mentioned above was noticed in its analytical aspect. The following is a geometrical proof of the same. We shall however, need the following Lemma at first.

Lemma.—In a triangle ABC, M,M′ are the middle points of the base BC and an intercept B′C′ parallel to BC. If the power of a point Z with respect to the circle described on XY as diameter be denoted by Z (XY), to show that |M (PB′) + M′ (PC) = M (PC′) + M′ (PB) = MM′ ^{2} where P is any point on the altitude from A.



Through C and B′ drop CX, B′X′ perp. to AM′M.

Since the triangles CMX and B′M′X′ are similar,

$$\frac{MX}{M'X'} = \frac{CM}{C'M'} = \frac{BM}{B'M'} = \frac{AM}{AM'}$$

$\therefore \qquad MX \,.\, M'A = M'X' \,.\, AM.$

$\therefore \qquad (M'X - M'M)\, M'A = (MX' - MM')\, AM.$

But $\qquad M\,(AC) = M'X \,.\, M'A$ and $M\,(AB') = -\, MX' \,.\, AM.$

$\therefore \quad M'\,(AC) + M\,(AB') = M'M\,(AM - AM') = M'M^2 \quad \ldots \quad (1)$

Again since, B′C′ is the radical axis of the circles (PB′) and (AB′),

$$M\,(PB') - M\,(AB') = 2\delta l$$

where δ is the distance between BC and B′C′ and l, the distance between the centres of the circles, and $AP = 2l$.

Hence, $\qquad M\,(PB') - M\,(AB') = \delta\, AP.$

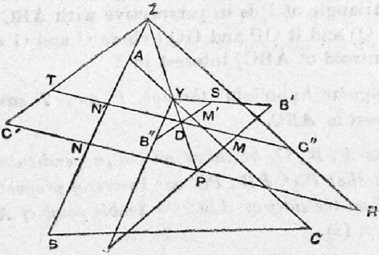
Similarly $\qquad M'\,(PC) - M'\,(AC) = (-\,\delta)\, AP.$

$\therefore \quad M\,(PB') + M'\,(PC) = M\,(AB') + M'\,(AC) = MM'^2. \qquad$ by (1)

Similarly, $\qquad M\,(PC') + M'\,(PB) = MM'^2.$

Hence the lemma is proved.

To revert to the original theorem, let YB″S and ZC′R be two triangles having the common median ZYDP ; also ZANB and AYMC be altitudes of ZC′R and YB″S, and TN′DC″ be parallel to C′R and B′MP parallel to B″S.



Now, in the triangle ZC′R, TDC″ is parallel to the base C′R and B is a point on the altitude from Z. Also, P is the mid point of CR and D that of TC″.

By the above lemma

$$P\,(BC'') + D\,(BC') = P\,D^2.$$

Similarly by considering the triangle Y B″S and the altitude Y M′C, we get

$$P\,(CB'') + D\,(CB') = PD^2.$$

$\therefore \quad P\,(BC'') + D\,(BC') = P\,(CB'') + D\,(CB'). \qquad \ldots (1)$

Now since when D is the double point of ABC and A'B'C', D lies on the radical axes of (BC') (CB'); (1) shows that P will lie on the radical axis of (BC") and (CB").

Remembering that D is the middle point of TC"

$$\frac{PC'}{PN} = \frac{DT}{DN'} = -\frac{DC''}{DN'} = v$$

since DT is parallel to PC'.

∴ Similarly
$$\frac{PB'}{PM} = \frac{DB''}{DM'} = \mu.$$

It follows therefore that if

$$(P, \lambda, \mu, v) = D,$$

then
$$(D, -\lambda, -\mu, -v) = P.$$

§ 6. *A few Miscellaneous Theorems.*

Before we pass on to the next part, it may be convenient to give without proof, a few theorems which incidentally arose during the progress of the present paper.

THEOREM IX.

If the pedal triangle of P is in perspective with ABC (the centre of perspective being Q) and if OP and GQ (where O and G are the circumcentre and the centroid of ABC) interest in T,

(i) *A rectangular hyperbola through P and T can be drawn self-conjugate with respect to ABC.*

(ii) *If points A', B', C' be taken on the perpendiculars from P on the sides of ABC such that PA', PB', PC' are inversely proportional to the perpendiculars from T on the sides of ABC, the double point of ABC and A'B'C' lies on the rect. hyp.* (i).

(iii) *The perpendiculars from A, B, C on the sides of A'B'C' concur at T.*

THEOREM X.

(i) *If A', B', C' be taken on the altitudes of a triangle such that the triangle A'B'C' is similar to the ortho-centric triangle, the locus of the double point of ABC and A'B'C' is the isogonal transformation of the Euler's line OH.*

(ii) *If the altitudes of a triangle ABC meet the circumcircle in A', B', C', the double point of ABC and A'B'C' is the isogonal conjugate of the nine-point centre.*

THEOREM XI.

Let DEF be the pedal triangle of P with respect to ABC. If points A′,B′,C′ are taken on the perpendiculars from P on the sides of ABC, such that

$$\frac{PA'}{PD} = \frac{PB'}{PE} = \frac{PC'}{PF} = \frac{RQ}{QP'}$$

where RQ is the diameter through P, of the circumcircle of ABC, the double point of ABC and A′B′C′ is at infinity (being the isogonal conjugate of Q); in other words, the mid-point lines of the quadrilaterals BCC′B′, CAA′C′, ABB′A′ are parallel to the pedal line of Q.

(To be concluded).

SHORT NOTES

An extended case of Hermite's Theorem

Hermite's theorem on partial fractions in Trigonometry states that

$$\frac{f(\sin\theta,\ \cos\theta)}{\prod_1^n \sin(\theta-\alpha_r)} = \sum_1^n \frac{f(\sin\alpha_r,\ \cos\alpha_r)}{P_r\,.\,\sin(\theta-\alpha_r)}$$

where $f(\sin\theta,\ \cos\theta)$ is a homogeneous function of the $(n-1)$th degree in $\sin\theta$, $\cos\theta$; and P_r denotes $\sin(\alpha_r-\alpha_1)\times\sin(\alpha_r-\alpha_2)\ldots\sin(\alpha_r-\alpha_n)$. This theorem is capable of extension as follows when the degree of f is still lower, say equal to m where $m<n$. Substituting x for $e^{i\theta}$, a_r for $e^{i\alpha_r}$, etc.

$$\frac{f_m(\sin\theta,\ \cos\theta)}{\prod_1^n \sin(\theta-\alpha_r)} = \frac{f_m\left(\dfrac{x^2-1}{2ix},\ \dfrac{x^2+1}{2x}\right)}{\dfrac{1}{(2i)^n}\prod_1^n\left(\dfrac{x}{a_r}-\dfrac{a_r}{x}\right)}$$

$$= (-)^n(2x)^{n-m}\frac{a_1a_2\ldots a_n f_m\left\{\dfrac{x^2-1}{i},\ x^2+1\right\}}{\prod_1^n (x^2-a_r^2)}$$

$$= (-)^n(2x)^{n-m}\frac{a_1a_2\ldots a_n \sum f_m\left[\dfrac{a_r^2-1}{i},\ a_r^2+1\right]}{(a_r^2-a_1^2)\ldots(a_r^2-a_n^2)}$$

$$= (-)^n\sum\left(\frac{x}{a}\right)^{n-m-1}\frac{f_m[(a_r^2-1)/2ia_r,\ (a_r^2+1)/2a_r]}{2^{-n}\prod_{s=1}^{n\,\prime}\left(\dfrac{a_r}{a_s}-\dfrac{a_s}{a_r}\right)(x_r^2-a_r^2)}$$

$$=\sum\frac{f_m(\sin\alpha_r,\ \cos\alpha_r)\,[\cos\overline{n-m-1}\,(\theta-\alpha_r)+i\sin\overline{n-m-1}(\theta-\alpha_r)]}{\sin(\theta-\alpha_r)\,.\,P_r}.$$

Hence

$$\frac{f_m(\sin\theta,\ \cos\theta)}{\prod_1^n \sin(\theta-\alpha_r)} = \sum_1^n\frac{f_m(\sin\alpha_r,\ \cos\alpha_r)\cos\overline{n-m-1}\,(\theta-\alpha_r)}{\sin(\theta-\alpha_r)\,.\,P_r}$$

which agrees of course, with Hermite's result when $m=n-1$.

G. S. M,

[The theorem is further capable of extension as follows :—

$$\frac{f_m(\sin\theta, \cos\theta)}{\prod_1^n \sin(\theta - \alpha_r)} = \sum_1^n \frac{f_m(\sin\alpha_r, \cos\alpha_r)}{P_r \sin(\theta - \alpha_r)},$$

$$\text{or} \quad = \sum_1^n \frac{f_m(\sin\alpha_r, \cos\alpha_r)}{P_r} \cot(\theta - \alpha_r),$$

according as $(n - m)$ is odd or even. To prove the latter case, which alone may be taken up in continuation of the above note, we observe that the left-hand member may be written $\dfrac{(\sin^2\theta + \cos^2\theta)^{\frac{n-m}{2}} f_m(\sin\theta, \cos\theta)}{\prod_1^n \sin(\theta - \alpha_r)}$

and can be expanded in a series of n co-tangents, viz., $\sum_1^n c_r \cot(\theta - \alpha_r)$,

because the resulting homogeneous identity is consistent. The value of c_r is obviously the value of $\left[\dfrac{f_m(\sin\theta, \cos\theta)\sin(\theta - \alpha_r)}{\prod \sin(\theta - \alpha_r)}\right]$ when $\theta = \alpha_r$;

or $c_r = \dfrac{f_m(\sin\alpha_r, \cos\alpha_r)}{P_r}$.

Reference may be made to the paper "On the factorization of $\dfrac{\sin mx}{\sin nx}$ and allied theorems" by Prof. A. C. L. Wilkinson (*J. I. M. S.*, Vol. I, pp. 72 *et seq*), and to Questions 82 and 104 by Mr. V. Ramaswami Aiyar in Vol. I of the *J. I. M. S.*—EDITOR.]

Aberration in Latitude and Longitude.

[Ref. In Ball's *Spherical Astronomy*, page 259, § 87 is considered the effect of the elliptic motion of the earth on aberration in R. A. and declination, but not on aberration in latitude and longitude. The expressions giving the aberration in these co ordinates are very simple unlike the lengthy ones for R. A. and declination. Remarks similar to those on pages 260, 261 apply to equations (3) and (4) below ; and examples 1 and 2 on page 261 are obvious corollories of the equations (3) and (4). It has therefore been considered useful to give these expressions in this note.]

§ 1. It is well known that the velocity of the earth at any instant can be resolved into two constant velocities :—

(i) $\frac{\mu}{h}$ perpendicular to the radius vector,

and (ii) $\frac{\mu e}{h}$ perpendicular to the major axis.

If $\odot$ = Longitude of the sun,

and ω = Longitude of perigee,

it is easily seen from a figure that the above velocitiees can be replaced by

$$\frac{\mu}{h}(\sin\odot + e\sin\omega) \qquad \text{along S} ♈.$$

$$-\frac{\mu}{h}(\cos\odot + e\cos\omega) \qquad \text{perpendicular to S} ♈.$$

If η_0 be the longitude of the earth's way, the velocity of the earth along the tangent to its path can be resolved into velocities

$v\cos\eta_0$ along S♈.

$v\sin\eta_0$ perpendicular to S♈;

so that we have

$$v\cos\eta_0 = \frac{\mu}{h}(\sin\odot + e\sin\omega) \qquad (1)$$

$$v\sin\eta_0 = -\frac{\mu}{h}(\cos\odot + e\cos\omega) \qquad (2)$$

§ 2. Now from (8) on page 252, Ball, we have if (λ, β) be the true longitude and latitude

(i) aberration in latitude $= -\frac{v}{u}\sin\beta\cos(\eta_0-\lambda)$

$$= -\frac{\mu}{hu}\sin\beta\,\{\sin(\odot-\lambda)+e\sin(\omega-\lambda)\}$$

where v is the velocity of the observer, and u that of light; (3)

(ii) aberration in longitude

$$= -\frac{v}{u}\sec\beta\sin(\lambda-\eta_0)$$

$$= -\frac{\mu}{hu}\sec\beta\,\{\cos(\odot-\lambda) + e\cos(\omega-\lambda)\} \qquad (4)$$

and $$\frac{\mu}{h} = \frac{2\pi a}{T\sqrt{1-e^2}} \text{ (Vide Q. 1004.)}$$

C. Krishnamachari.

It will easily be verified that

$$\frac{d}{dx}\log\begin{bmatrix}\sin x\\ \operatorname{cosec} x\end{bmatrix} = \pm\cot x,$$

$$\frac{d}{dx}\log\begin{bmatrix}\cos x\\ \sec x\end{bmatrix} = \mp\tan x;$$

and $$\frac{d}{dx}\log\begin{bmatrix}\tan x\\ \cot x\end{bmatrix} = \pm 2\operatorname{cosec} 2x;$$

so that $$\left(\frac{d}{dx}\log\right)\left(\frac{d}{dx}\log\right)\begin{bmatrix}\sin x\\ \operatorname{cosec} x\end{bmatrix} = -2\operatorname{cosec} 2x, \text{ etc.};$$

and $$\left(\frac{d}{dx}\log\right)\left(\frac{d}{dx}\log\right)\left(\frac{d}{dx}\log\right)\begin{bmatrix}\sin x\\ \operatorname{cosec} x\end{bmatrix} = -2\cot 2x.$$

In general, if θ denote the logarithmic derivate, viz., the operation $\left(\frac{d}{dx}\log\right)$; and $n > 1$.

$$\theta^{2n}(\sin x) = \theta^{2n}(\cos x) = \theta^{2n}(\operatorname{cosec} x) = \theta^{2n}(\sec x)$$
$$= -2^n\operatorname{cosec}(2^n x);$$
$$\theta^{2n}(\tan x) = \theta^{2n}(\cot x) = -2^n\cot 2^n x;$$
$$\theta^{2n+1}(\sin x) = \theta^{2n+1}(\cos x) = \theta^{2n+1}(\operatorname{cosec} x) = \theta^{2n+1}(\sec x)$$
$$= -2^n\cot 2^n x;$$
$$\theta^{2n+1}(\tan x) = \theta^{2n+1}(\cot x) = -2^{n+1}\operatorname{cosec}(2^{n+1}x).$$

* * * * *

It is perhaps well known that

The locus of the midpoint of the intercept of any tangent to a parabola on two fixed tangents is a tangent to the parabola (at the point whose parameter is the mean of the parameters of the points of contact of the fixed tangents).

If t_1, t_2 and t are respectively the parameters and X_1, X_2, the points where (t_1, t) (t_2, t) meet, the middle point of X_1X_2 will be

$$\left(at\,\overline{\frac{t_1+t_2}{2}};\ at + \frac{a}{2}\,\overline{t_1+t_2}\right)$$

whose locus is obviously

$$y = \frac{a}{2}(t_1 + t_2) + x\,\frac{2}{t_1 + t_2},$$

which demonstrates the theorem.

This theorem can be extended as follows :

The locus of the mean centre of the n points of intersection of any tangent to a parabola with n fixed tangents is also a tangent (at the point whose parameter is the mean of those of the n fixed points of contact).

With obvious notation, X_r is $(a\,t\,t_r, a\,\overline{t + t_r})$, and the mean centre of $X_1 \ldots\ldots X_n$ is $\xi = at\,\frac{\Sigma t_r}{n}$, $\eta = at + a\,\frac{\Sigma t_r}{n}$.

Hence the locus of the mean centre is

$$y = a\,\Sigma(t^r{}_{,}/n + an/\Sigma(t_r)$$

which is the tangent at the point whose parameter is the artihemetic mean of those of the fixed points.

It will also be recognised that P is the point dividing in the ratio $1 : n - 1$ the line joining the mean centre of the points of contact of the n fixed tangents with the mean centre of their $n(n - 1)/2$ intersections.

SOLUTIONS TO QUESTIONS.

Question 791.

(K. Appukuttan Erady, M.A.) The centres of three circles of radii a, b, c form a triangle of sides l, m, n and area $\triangle$. If (r_1, r_1'), (r_2, r_2') (r_3, r_3') and (r_4, r_4') be the radii of the four pairs of circles tangential to the three circles (the circles belonging to any pair being inverses of each other with respect to the common orthogonal circle of the three original circles), show that $\frac{1}{r_1 r_1'} + \frac{1}{r_2 r_2'} + \frac{1}{r_3 r_3'} + \frac{1}{r_4 r_4'}$

$$= \frac{16 (\triangle^2 - \Sigma a^2 l^2)}{\Sigma l^2 (a^2-b^2)(a^2-c^2) - \Sigma m^2 n^2 (b^2 + c^2) + \Sigma a^2 l^4 + l^2 m^2 n^2}$$

Solution by N. Sankara Aiyar and Martyn M. Thomas.

The numerator should be $16 (4 \triangle^2 - \Sigma a^2 l^2)$ as can be seen by making $a = b = c = 0$.

Let LMN be the triangle of centres, P the centre of the circle touching the three given circles internally and r_1 its radius.

Then $PL = r_1 + a$, $PM = r_1 + b$, $PN = r_1 + c$.

If $P\hat{L}M = \theta$ and $N\hat{L}M = A$, we get

$$(r_1 + b)^2 = n^2 + (r_1 + a)^2 - 2n (r_1 + a) \cos \theta$$

and $$(r_1 + c)^2 = l^2 m^2 + (r_1 + a)^2 - 2m (r_1 + a) \cos \overline{A - \theta}$$

or, $$2n (r_1 + a) \cos \theta = 2 r_1 \overline{a - b} + n^2 + a^2 - b^2 = h \text{ (say)}$$

and $$2m (r_1 + a) \cos (A - \theta) = 2r_1 \overline{a - c} + m^2 + a^2 - c^2 = k \text{ (say)}$$

$$\therefore \quad 4mn (r_1 + a) \sin \frac{A}{2} \sin \left(- \theta \frac{A}{2} \right) = nk - mh,$$

and $$4mn (r_1 + a) \cos \frac{A}{2} \cos \left(\frac{A}{2} - \theta \right) = nk + mh.$$

Multiplying by $\cos \frac{A}{2}$, $\sin \frac{A}{2}$, squaring and adding

$$16 m^2 n^2 (r_1 + a)^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2} = \cos^2 \frac{A}{2} (nk - mh)^2 + \sin^2 \frac{A}{2} (nk + mh)^2 .$$

$$\therefore\ 4m^2n^2(r_1+a)^2\sin^2 A = \frac{1+\cos A}{2}(nk-mh) + \frac{1-\cos A}{2}(nk+mh)^2$$

$$= n^2k^2 + m^2h^2 - 2mnkh\cos A$$

$$= n^2\{2r_1\overline{a-c} + m^2 + a^2 - c^2\}^2$$

$$+ m^2\{2r_1\overline{a-b} + n^2 + a^2 - b^2\}^2$$

$$-2mn\cos A\{2r_1\overline{a-c} + m^2 + a^2 - c^2\}$$

$$\{2r_1\overline{a-b} + n^2 + a^2 - b^2\}.$$

Expanding and writing down the term containing r_1^2 and the constant term we get

$$r_1^2\{4m^2n^2\sin^2 A - 4n^2(a-c)^2 - 4m^2(a-b)^2 + 8mn\cos A$$

$$(a-c)(a-b)\}$$

$$+ 4m^2n^2a^2\sin^2 A - n^2(m^2+a^2-c^2)^2 - m^2(n^2+a^2-b^2)^2$$

$$+ 2mn\operatorname{Cos} A(m^2+a^2-c^2)(n^2+a^2-b^2) + r_1\{\ldots\ldots\ldots\} = 0.$$

$$\therefore\ r_1^2\{16\Delta^2 - 4n^2(a-c^2) - 4m^2(a-b)^2 + 4(m^2+n^2-l^2)$$

$$(a-c)(a-b)\}$$

$$+ 4m^2n^2a^2\sin^2 A - n^2m^4 - 2n^2m^2(a^2-c^2) - (a^2-c^2)^2 - m^2n^4$$

$$-2m^2n^2(a^2-b^2) - (a^2-b^2)^2 + (m^2+n^2-l^2)(m^2+a^2-c^2)$$

$$(n^2+a^2-b^2)$$

$$+ r_1\{\ldots\ldots\ldots\} = 0,$$

which becomes after some reductions

$$r_1^2\{16\Delta^2 - 4\Sigma a^2l^2 + 4\Sigma bc(l^2+m^2-n^2)\}$$

$$-l^2m^2n^2 + \Sigma m^2n^2(b^2+c^2) - \Sigma a^2l^4 - \Sigma l^2(a^2-b^2)(a^2-c^2)$$

$$+ r_1\{\ldots\ldots\ldots\ldots\} = 0. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \text{ (i)}$$

The coefficient of r_1 being an odd function of a,b,c will change sign when a,b,c all change sign, *i.e.*, the equation for radius touching all internally will be the same as (i) with the sign of the middle term changed.

Hence the value of $\dfrac{1}{r_1 r_1'}$ is seen to be

$$\frac{16\Delta^2 - 4\Sigma a^2l^2 + 4\Sigma bc(l^2+m^2-n^2)}{\Sigma l^2(a^2-b^2)(a^2-c^2) - \Sigma m^2n^2(b^2+c^2) + \Sigma a^2l^4 + l^2m^2n^2}.$$

Changing the signs of a,b,c one at a time, we get the values of

$$\frac{1}{r_2 r_2'},\ \frac{1}{r_3 r_3'},\ \frac{1}{r_4 r_4'};$$

adding we get $\frac{1}{r_1 r_1'} + \frac{1}{r_2 r_2'} + \frac{1}{r_3 r_3'} + \frac{1}{r_4 r_4'} =$

$$\frac{64\Delta^2 - 16 \Sigma a^2 l^2 + 4 \Sigma (l^2 + m^2 - n^2)(bc - bc - bc + bc)}{\Sigma l^2 (a^2 - b^2)(a^2 - c^2) - \Sigma m^2 n^2 (b^2 + c^2) + \Sigma a^2 l^4 + l^2 m^2 n^2},$$

which gives the required result.

Question 839

(R. VYTHYNATHASWAMY) :—In any system of parametric representation of the nodal cubic, the parameters of the points of inflexion are given to be the roots of the equation

$$At^3 + 3\,Bt^2 + 3\,Ct + D = 0.$$

Prove that the condition of collinearity of the points t_1, t_2, t_3 is

$$At_1t_2t_3 + B\,\Sigma t_1t_2 + C\,\Sigma t_1 + D = 0,$$

and that the parameters of the nodal points are the roots of the equation

$$(AC - B^2)\,x^2 + (AD - BC)\,x + BD - C^2 = 0.$$

Solution by Hemraj

If a cubic has a double point, the co-ordinates are rational functions of one parameter. We may start with the expressions for the co-ordinates in terms of a parameter t, viz.

$$x = at^3 + 3bt^2 + 3\,c\,t + d,$$

$$y = a_1\,t^3 + 3b_1\,t^2 + 3c_1\,t + d_1,$$

$$z = a_2\,t^3 + 3b_2\,t^2 + 3c_2\,t + d_2.$$

$\therefore \quad \lambda x + \mu y + \nu z = 0$ must always have three roots.

The condition that three points t_1, t_2, t_3 of the curve should be collinear is

$$\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} = 0.$$

This is easily seen to be resolvable into partial determinants each of which is divisible by $(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$ and the condition is

$$At_1t_2t_3 + B\,\Sigma\,t_1t_2 + C\,\Sigma\,t_1 + D = 0, \quad \dots \quad \dots (1)$$

A, B, C, D being the determinants of the system

$$\begin{Vmatrix} a & b & c & d \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{Vmatrix}$$

Given t_1, t_2, it usually determines t_3 uniquely, but if

$$A\,t_1t_2 + C\,(t_1 + t_2) + C = 0,$$

and
$$B\,t_1t_2 + C\,(t_1 + t_2) + D = 0,$$

it is identical in t_3. In this case t_1, t_2 are the roots of

$$(At + B)\,(Ct + D) - (Bt + C)^2 = 0,$$

i.e.
$$(AC - B^2)\,t^2 + (AD - BC)\,t + BD - C^2 = 0. \qquad (2)$$

These values of t belong to the same point on the curve which is, therefore, a double point, node, cusp, or conjugate point according to circumstances.

If t is an inflexion, t,t,t are collinear, and from (1), t is given by

$$At^3 + 3\,Bt^2 + 3\,Ct + D = 0. \qquad (3)$$

There are three points of inflexions and any cubic gives collinear points, if it is a polar to the cubic giving the inflexions.

If three points be determined by the cubic

$$A't^3 + 3\,B't^2 + 3\,C't + D' = 0,$$

then the condition that three points should be on a right line is

$$(AD' - A'D) - 3\,(BC' - B'C) = 0, \qquad \text{from (1)}.$$

Hence the three inflexions are collinear.

The quadratic (2) giving the double point is the Hessian of the cubic giving the flexes.

Thus if a cubic has a crunode, it has one real point of inflexion.

If a cubic has an acnode, it has three real points of inflexion.

If a cubic has a spinode, the roots of the Hessian are equal, and the cubic has the same two roots equal corresponding to the cusp, and the third real root gives the inflexion. Hence a cuspidal cubic has only one flex and that must be real.

Question 940.

(R. Srinivasan, M.A., M.R.A.S.) :—Show that

$$\int_0^1 \frac{x^n \log x}{1 + x^2}\,dx = \frac{\pi^2}{4} \sec \frac{n\pi}{2} \tan \frac{n\pi}{2}.$$

Remarks by K. J. Sanjana, M.A.

The result is evidently inaccurate. If n is positive, the function $x^n/(1 + x^2)$ is positive and finite between the limits of integration. Hence the integrand is finite and negative throughout, since both at $x = 1$ and $x = 0$, $x^n \log x$ vanishes. But the right-hand side is positive from $n = 0$ to $n = 1$, and at $n = 1$ it is infinitely large.

Let $f(n) = \int_0^1 \frac{x^n dx}{1 + x^2}$; then $f'(n) = \int_0^1 \frac{x^n \log x}{1 + x^2} dx$.

Also $f(n) = \int_0^1 x^n(1 - x^2 + x^4 - \ldots) \, dx$, as the expansion is valid,

$$= \frac{1}{n+1} - \frac{1}{n+3} + \frac{1}{n+5} - \ldots\ldots$$

Hence the integral $f'(n) =$

$$- \left[\frac{1}{(n+1)^2} - \frac{1}{(n+3)^2} + \frac{1}{(n+5)^2} - \ldots \textit{ ad inf.}\right]$$

The result may be obtained directly from the formula

$$\int_0^1 x^r \log x \, dx = -(r+1)^{-2}.$$

No finite form can be given to the sum except for odd integral positive values of n; for even integral values the sum has been calculated numerically by Dr. Glaisher and others.

Question 955.

(K. B. Madhava):—The 5th, 13th, 17th, 29th and 37th roots (if rational) of a number end in the same digit as the number itself, the numbers when fractional being expressed in decimals. Prove this property and illustrate how it enables one to give out at sight the above rational roots of big numbers.

Solution by Prof. N. B. Mitra, M.A.

Let all the letters in this solution, except N and n denote positive integers and let N and n denote any positive rational quantities.

Let d stand for any of the ten digits 0, 1, 2, ……, 9; and k for any of the four digits 1, 3, 7, 9.

Further let all the congruences in this solution be understood to be to the modulus 10.

Also, if a rational fraction converted to decimals be terminating then by the last digit of the fraction we shall mean the units figure of the number obtained by removing the decimal point from the equivalent decimal fraction written in its simplest form. Thus although $\cdot 456 = \cdot 4560$, by the last figure of this fraction we shall mean 6. Similarly by the last digit of a recurring decimal like $\cdot 45\dot{6}\dot{7}$, we shall mean 7, although $\cdot 45\dot{6}\dot{7} = \cdot 456\dot{7}\dot{5} = \cdot 4567\dot{5}\dot{6}$, &c.

Now $0^q \equiv 0$, $1^q \equiv 1$, $5^q \equiv 5$, $6^q \equiv 6$:

$\therefore$ if $d = 0, 1, 5$ or 6, $d^2 \equiv d$. ... ... (*a*)

Again $4^2 \equiv 6$; $4^{2q} \equiv 6^q \equiv 6$, $\therefore$ $4^{2q+1} \equiv 24 \equiv 4$;

also $9^2 \equiv 1$, $\therefore$ $2^{2q} \equiv 1$ and $9^{2q+1} \equiv 9$.

$\therefore$ if $d = 4$ or 9, $d^{2q+1} \equiv d$.... ... ... (*b*)

Lastly, $2^4 \equiv 6$, $\therefore$ $2^{4q} \equiv 6^q \equiv 6$; $\therefore$ $2^{4q+1} \equiv 12 \equiv 2$;

$3^4 \equiv 1$, $\therefore$ $3^{4q} \equiv 1$ and $3^{4q+1} \equiv 3$;

$7^4 \equiv 1$, $\therefore$ $7^{4q+1} \equiv 7$;

and $8^4 \equiv 6$, $\therefore$ $8^{4q+1} \equiv 8$.

$\therefore$ if $d = 2, 3, 7$ or 8, $d^{4q+1} \equiv d$ ... ... (*c*)

Hence from (*a*), (*b*) and (*c*), we see that in all cases $d^{4q+1} \equiv d$ and $k^{4q} \equiv 1$; in other words, the congruence $d^p \equiv d$ is satisfied if p is of the form $4q + 1$. ... ... (A)

Further the congruence $k^p \equiv 1$ is satisfied if p is of the form $4q$.

Now let $\sqrt[p]{N} = n$, so that $N = n^p$.

Two cases arise:

First: let n be an integer and let $n = d$ $(d < 10)$.

$\therefore$ $N = n^p \equiv d^p$.

Hence N and n will end in the same digit if $d^p \equiv d$, *i.e.*, if p is of the form $4q + 1$ by (A).

Next: let n be a fraction in its lowest terms and $= \frac{a}{b}$, so that a and b are relative primes.

Let $b = 2^l 5^m c$, where c is prime to 10, so that $c \equiv k$. ... (1)

Let f stand for either 2 or 5, the factors of 10 and let $l \sim m = g$.

Also let h stand for either l or m.

Now $n = \dfrac{a}{2^l 5^m c} = \dfrac{af^g}{10^h c}$.

Let $cx \equiv 9$ $(x < 10)$ ... ... (2)

and $af^g \equiv d$ $(d < 10)$ ... ... (3)

Then the last figure of n which equals $\dfrac{af^g x}{10^h cx}$ is the same as that of $af^g x$, *i.e.*, of dx.

Again $N = \dfrac{a^p f^{gp}}{10^{hp} c^p}$. Let $c^p y \equiv 9$ $y < 10)$ ... ... (4)

Then the last figure of N is the same as that of a^p for y, *i.e.*, of d^py by (3).

Hence n and N will end in the same figure if $dx \equiv d^py$, *i.e.*, if $dxc^p \equiv d^pyc^p$, *i.e.*, if $dc^{p-1}9 \equiv d^p 9$ by (2) & (4) *i.e.*, if $dc^{p-1} \equiv d^p$, *i.e.*, if $dk^{p-1} \equiv d^p$ by (1) ... ... (B)

From (A) and (B), we see that this is satisfied if p is of the form $4q+1$.

Hence, with the understanding set forth in the beginning, any number, integral or fractional and its $(4q+1)$ th root (when rational) will end in the same digit, q being any positive integer.

Partial Solutions by Sadanand and C. Krishnamachari.

Question 961.

(H. Br.) :—Show that

$$x^2 = y^2 + z^2, \quad t^2 = y^2 + 4z^2$$

have no solutions in non-zero integers.

Solution by N. Sankara Aiyar.

We require that y can be expressed as $m^2 - n^2$ and $M^2 - N^2$ and z as MN and $2mn$.

Therefore we may suppose that since y and z have no common factors unless they are factors of x and t also, the same is true of m, n, M, N.

$$\text{Let} \quad \left.\begin{aligned} m &= ab, \ M = ac \ \text{ or } \ bc \\ n &= cd, \ N = 2bd \ \text{ or } \ 2ad \end{aligned}\right\}$$

where a, b, c, d are prime to one another and d is even.

(1) If M $= ac$, we get

$$a^2b^2 - c^2d^2 = a^2c^2 - 4b^2d^2$$

i.e. $$b^2(a^2 + 4d^2) = c^2(a^2 + d^2)$$

i.e. $$\frac{a^2 + 4d^2}{c^2} = \frac{a^2 + d^2}{b^2} = v \text{ (say)} = \frac{3d^2}{c^2 - b^2}.$$

Hence if $v = 3$, d is divisible by v and so also is a since $a^2 + d^2 = vb^2$.

Therefore $v = 1$, we get $a^2 + 4d^2 = c^2$ and $a^2 + d^2 = b^2$, a simple set of solution for the original equation where a, b, c, d are less than y, x, z, t.

If $v = 3$, then $d^2 = c^2 - b^2$, and $a^2 + c^2 - b^2 = 3b^2$,

i.e. $$a^2 + c^2 = 4b^2$$

which cannot be true unless both a and c are even.

(2) If $M = bc$, we get

$$a^2b^2 - c^2d^2 = b^2c^2 - 4a^2d^2$$

i.e. $$a^2(b^2 + 4d^2) = c^2(b^2 + d^2)$$

i.e. $$\frac{b^2 + 4d^2}{c^2} = \frac{b^2 + d^2}{a^2} = v \text{ (say)} = \frac{3d^2}{c^2 - a^2};$$

and as before this reduces to $b^2 + 4d^2 = c^2$, $b^2 + d^2 = a^2$.

In either case, we have to solve the equation for smaller values of x, y, z, t. This can be carried on till $y = 1$.

We get $x^2 = z^2 + 1$, $t^2 = 4z^2 + 1$, which is impossible.

Question 973.

(A. C. L. WILKINSON) :—(i) Prove that

$$D \operatorname{sn}(u_1 + u_2 + u_3) = \begin{vmatrix} s_1, & s_1', & s_1^3 \\ s_2, & s_2', & s_2^3 \\ s_3, & s_3', & s_3^3 \end{vmatrix},$$

$$D \operatorname{cn}(u_1 + u_2 + u_3) = \begin{vmatrix} c_1, & c_1', & c_1^3 \\ c_2, & c_2', & c_2^3 \\ c_3, & c_3', & c_3^3 \end{vmatrix},$$

$$k^4 \operatorname{dn}(u_1 + u_2 + u_3) = -\begin{vmatrix} d_1, & d_1', & d_1^3 \\ d_2, & d_2', & d_2^3 \\ d_3, & d_3', & d_3^3 \end{vmatrix}$$

where $$D = \begin{vmatrix} 1, & s_1^2, & s_1 c_1 d_1 \\ 1, & s_2^2, & s_2 c_2 d_2 \\ 1, & s_3^2, & s_3 c_3 d_3 \end{vmatrix},$$ and $s_1 = \operatorname{sn} u_1$ and $s_1' = \dfrac{d}{du_1}(\operatorname{sn} u_1)$ etc.

(ii) Prove that

$$E(u_1) + E(u_2) + E(u_3) + E(u_4) - 2E(x)$$

$$= \frac{k^2 d_1 d_2 d_3 d_4}{k_1^2 + d_1 d_2 d_3 d_4}\left(\frac{s_1 c_1}{d_1} + \frac{s_2 c_2}{d_2} + \frac{s_3 c_3}{d_3} + \frac{s_4 c_4}{d_4}\right)$$

$$- \frac{2k^2 \operatorname{sn} x \operatorname{cn} x \operatorname{dn}(x - u_3 - u_4)\operatorname{dn}(x - u_2 - u_4)\operatorname{dn}(x - u_2 - u_3)}{k_1^2 + \operatorname{dn} x \operatorname{dn}(x - u_3 - u_4)\operatorname{dn}(x - u_2 - u_4)\operatorname{dn}(x - u_2 - u_3)}$$

where $x = \frac{1}{2}(u_1 + u_2 + u_3 + u_4)$ and s_1, c_1, d_1 are written as usual for $\operatorname{sn} u_1$, $\operatorname{cn} u_1$, $\operatorname{dn} u_1$

Solution by F. H. V. Gulasekharam.

(i) The results cited here for sn, cn, dn of $(u_1 + u_2 + u_3)$ are identically the same as mine published in J. I. M. S, Vol. VIII, p. 211. Professor Wilkinson expresses them in an elegant determinant form.

(ii) If $u_1 + u_2 + u_3 + u_4 = 2x$, we can easily prove

$$(a) \quad \frac{1}{k'^2} \operatorname{dn} x \operatorname{dn}(x - u_3 - u_2) \operatorname{dn}(x - u_2 - u_4) \operatorname{dn}(x - u_3 - u_4)$$

$$= \frac{k'^2 + k^2 k'^2 s_1 s_2 s_3 s_4 + k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4}{k'^2 - k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4}.$$

$$\therefore \quad \frac{1}{k'^2}(k'^2 - k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4)$$

$$= \frac{2(k'^2 + d_1 d_2 d_3 d_4)}{k'^2 + \operatorname{dn} x \operatorname{dn}(x - u_3 - u_2) \operatorname{dn}(x - u_2 - u_4) \operatorname{dn}(x - u_3 - u_4)}.$$

$$(b) \quad \frac{\Theta(x).\ \Theta(x - u_3 - u_2).\ \Theta(x - u_2 - u_4).\ \Theta(x - u_3 - u_4)}{\Theta(u_1).\ \Theta(u_2).\ \Theta(u_3).\ \Theta(u_4)}$$

$$= \frac{1}{k'^2}(k^2 - k^2 k'^2 s_1 s_2 s_3 s_4 - k^2 c_1 c_2 c_3 c_4 + d_1 d_2 d_3 d_4)$$

$$= \frac{2 . k'^2 + d_1 d_2 d_3 d_4)}{k'^2 + \operatorname{dn} x \operatorname{dn}(x - u_3 - u_2) \operatorname{dn}(x - u_2\ u_4) \operatorname{dn}(x - u_3 - u_4)}.$$

where $\Theta(u)$ stands for the Jacobian Theta fraction of u.

[*Vide*: Whitaker and Watson's *Modern Analysis*, Revised Edition, pp. 460-461].

[Now differentiating logarithmically with respect to u_1 both sides of the identity (b) and treating u_2, u_3, u_4 as constants, and remembering that

$$\frac{dx}{du_1} = \tfrac{1}{2}, \text{ and } \frac{\Theta'(u)}{\Theta(u)} = \mathrm{E}(u) - u\frac{\mathrm{E}}{\mathrm{K}}, \text{ we get}$$

$$\mathrm{E}(x) + \mathrm{E}(x - u_3 - u_2) + \mathrm{E}(x - u_2 - u_4) + \mathrm{E}(x - u_3 - u_4) - 2\,\mathrm{E}(u_1)$$

$$= \frac{k^2 \operatorname{dn} x \operatorname{dn}(x - u_3 - u_2) \operatorname{dn}(x - u_2 - u_4) \operatorname{dn}(x - u_3 - u_4)}{k'^2 + \operatorname{dn} x \operatorname{dn}(x - u_3 - u_2) \operatorname{dn}(x - u_2 - u_4) \operatorname{dn}(x - u_3 - u_4)}$$

$$\left[\frac{\operatorname{sn} x \operatorname{cn} x}{\operatorname{dn} x} + \frac{\operatorname{sn}(x - u_3 - u_2) \operatorname{cn}(x - u_3 - u_2)}{\operatorname{dn}(x - u_3 - u_2)}\right.$$

$$+\frac{\text{sn}(x-u_2-u_4)\,\text{cn}(x-u_2-u_4)}{\text{dn}(x-u_2-u_4)}+\frac{\text{sn}(x-u_3-u_4)\,\text{cn}(x-u_3-u_4)}{\text{dn}(x-u_3-u_4)}\Bigg]$$

$$-\frac{k^2 s_1 c_1 d_2 d_3 d_4}{k'^2+d_1 d_2 d_3 d_4}.$$

Since it is obvious that in the above, x, $x-u_3-u_2$, $x-u_2-u_4$, $x-u_3-u_4$ are interchangeable with u_1, u_2, u_3, u_4 respectively, we get the required result.

Question 985.

(M. K. KEWALRAMANI) :—If A′, B′, C′ are points in the sides of the triangle ABC such that BA′ : A′C = CB′ : B′A = AC′ : C′B = M : 1, find an expression for the radius of the circumcircle of the triangle XYZ formed by the intersections of AA′, BB′, CC′.

Solution by N. B. Mitra, Sadanand, and G. V. Krishnasawmi.

Taking ABC for the triangle of reference, the equations of AA′, BB′, CC′ are

$$\begin{aligned} Mb\beta - c\gamma &= 0 \\ -a\alpha + Mc\gamma &= 0 \\ Ma\alpha - b\beta &= 0. \end{aligned}$$

Hence the co-ordinates of Z are given by

$$a\alpha : b\beta : c\gamma = 1 : M : M^{-1},$$

and so for X and Y.

Now the area of the triangle whose vertices are $(\alpha_1, \beta_1, \gamma_1)$, &c., is

$$\frac{abc}{8\Delta}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$\therefore \text{ area of XYZ} = \frac{abc}{8\Delta^2}\cdot\frac{8\Delta_3}{(M^2+M+1)^3\,abc}\begin{vmatrix} 1 & M & M^2 \\ M & M^2 & 1 \\ M^2 & 1 & M \end{vmatrix}$$

$$= \frac{\Delta(M-1)^2}{1+M+M^2} \text{ (without regard to the sign.)}$$

Again by the formula for the sine of the angle between two lines,

$$\sin X = 2\Delta(1+M+M_2) \div \sqrt{\{(a^2+M^2c^2+2Mac\cos B)(b^2+M^2a^2+2Mab\cos C)\}}.$$

Similarly for sin Y & sin Z.

Hence the circum-radius of XYZ is

$$\sqrt{\frac{\text{area of XYZ}}{2\sin X\sin Y\sin Z}}$$

$$= \frac{M-1}{4\Delta(1+M+M^2)^2}\{+a^2+M^2c^2+2Mac\cos B)(b^2+M^2a^2+2abM\cos C)\times(c^2+M^2b^2+2Mbc\cos A)\}^{\frac{1}{2}}.$$

NOTE.—It can be shown that circum-radius of XYZ

$$= \frac{abc}{8\Delta}\,\frac{\sin(2A+\beta-\gamma)\cos\alpha+\sin(2B+\gamma-\alpha)\cos\beta+\sin(2C+\alpha-\beta)\cos\gamma}{\sin(A+\beta-\gamma)\sin(B+\gamma-\alpha)\sin(C+\alpha-\beta)}$$

where $\cot\alpha = \frac{\cot B - M\cot C}{1+M}$, $\cot\beta = \frac{\cot C - M\cot A}{1+M}$,

& $\cot\gamma = \frac{\cot A - M\cot B}{1+M}$.

Question 988.

(S. MALHARI RAO):--Complete the following magic square of 16 cells, by placing 8 other prime numbers in the vacant cells :

7	43	71	113
73	97	23	41

Solution by K. B Madhava, Jamuna Prasad Nigama, R. D. Karve and M. K. Kewalramani.

The following *three* squares satisfy the problem proposed :

7	43	71	113
A	83	37	B
C	11	103	D
73	97	23	41

where the corresponding values of A, B, C and D are

A = 101, 53, 47 ; B = 13, 61, 67
C = 53, 101, 107 ; D = 67, 19. 13,

Question 989.

(S. Malhari Rao) :—Give three solutions in positive integers of the equation $(x + y + z)^3 = xyz$.

Solution by N. B. Mitra, N. B. Pendse, and Sadanand.

Assume $x : y : z :: l : m : n$.

Then substituting in the proposed equation, we find

$$x = \frac{(l + m + n)^2}{mn}, \quad y = \frac{(l + m + n)^2}{nl}, \quad z = \frac{(l + m + n)^2}{lm}.$$

This is the most general solution in *rational* numbers, not necessarily integral.

To get integral solutions, we give particular values to two of the quantities l,m,n and determine the values of the third so that x,y,z may, if possible, be integral.

Thus if $l = 1$, $m = 1$, then $x = y = n + 4 + \frac{4}{n}$, so that, we may take $n = 1, 2$, or 4; and the corresponding solutions of (1) are

(i) 9,9,9. (ii) 8,8,16. (iii) 9,9,36.

If $l = 1$, $m = 2$, $x = \frac{1}{2}\left(x + 6 + \frac{9}{n}\right)$ and $\therefore$ $n = 1$, 3 or 9, giving the solutions

(ii) 8,16,8. (iv) 6,12,18 (v) 8,16,72.

If $l = 1$, $m = 3$, n must be 2 or 8 and the solutions are

(iv) 6,18,12. (vi) 6,18,48.

It may be observed that, if $l = 1$, then $(m + 1)^2 \equiv 0$, (mod. n) and in addition $n + 1 \equiv 0$, (mod. m), when m is a prime or the square of a prime.

Some more solutions together with the corresponding values of l, m & n are given below :—

	(vii)	(viii)	(ix)	(x)	(xi)	(xii)	(xiii)	(xiv)	(xv)	(xvi)	(xvii)
$l =$	1,	1,	1,	1,	1,	1,	3,	4,	5,	2,	2,
$m =$	4,	4,	5,	8,	9,	9,	8,	5,	9,	3,	9,
$n =$	5,	25,	9,	27,	20,	50,	121,	81,	196,	25,	121,
$x =$	5,	9,	5,	6,	5,	8,	18,	20,	25,	12,	16,
$y =$	20,	36,	25,	48,	45,	75,	48,	25,	45,	18,	72,
$z =$	25,	225,	45,	162,	100,	400,	726,	405,	980,	150,	968,

An indefinite number of solutions may thus be obtained.

Question 993.

(Prof. Hemraj) :—From a point P perpendiculars PL, PM, PN are drawn to the sides of a triangle ABC. If AL, BM, CN meet at the point Q, find the locus of P, and prove that PQ passes through a fixed point.

Solution by Martyn M. Thomas.

Taking ABC as the triangle of reference, let P be (α,β,γ) and Q (l,m,n) so that eqn. to AQL is $\dfrac{y}{z} = \dfrac{m}{n}$. $\therefore$ L is $(o,m,n,)$ and eqn. to PL is

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ o & m & n \end{vmatrix} = 0. \quad \therefore x(\beta n - \gamma m) - y\alpha n + z\alpha m = 0.$$

The condition that PL may be perpendicular to BC is therefore

$$-m(\alpha\cos B + \gamma) + n(\alpha\cos C + \beta) = 0. \quad \dots \quad \dots (1)$$

Similarly, conditions of perpendicularity for (PM, AC) and (PN, AB) are

$$l(\beta\cos A + \gamma) - n(\beta\cos C + \alpha) = 0, \quad \dots (2)$$

$$-l(\gamma\cos A + \beta) + m(\gamma\cos B + \alpha) = 0. \quad \dots (3)$$

Eliminating l,m,n from (1), (2) and (3), the locus of P is

$$\begin{vmatrix} o, & -(\alpha\cos B + \gamma), & (\alpha\cos C + \beta) \\ \beta\cos A + \gamma, & o, & -(\beta\cos C + \alpha) \\ -(\gamma\cos A + \beta), & \gamma\cos B + \alpha, & o \end{vmatrix} = 0.$$

$$i.e. \left(\cos A + \frac{\gamma}{\beta}\right)\left(\cos B + \frac{\alpha}{\gamma}\right)\left(\cos C + \frac{\beta}{\alpha}\right) - \left(\cos A + \frac{\beta}{\gamma}\right)\left(\cos B + \frac{\gamma}{\alpha}\right)\left(\cos C + \frac{\alpha}{\beta}\right) = o$$

$$i.e. \sum\left\{\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)\left(\cos C - \cos A\cos B\right)\right\} = 0.$$

This locus has been verified to pass through the in-centre (1,1,1), the three ex-centres, (−1,1,1), (1,−1,1), (1,1,−1) the circum-centre (cos A, cos B, cos C) and the ortho-centre (sec A, sec B, sec C).

Equation to the line PQ is

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ l & m & n \end{vmatrix} = 0,$$

which is seen to pass through the fixed point

(cos A — cos B cos C) : (cos B — cos C cos A) : cos C — cos A cso B).

For, if we multiply (1), (2), (3) by cos A, cos B, cos C and add, we have Σ [cos A — cos B cos C) $(\beta n - \gamma m,] = 0$, whence the result follows.

Question 995.

(T. P. Trivedi, M.A., LL.B):—Prove that

$$\int_0^1 x^{x^x} dx = \sum_{n=0}^{n=\infty} (-1)^n \left\{ \frac{1^n}{1^{n+1}} - n.\frac{n+1}{2^{n+2}} + \frac{n^2}{2!}.\frac{(n+1)(n+2)}{3^{n+3}} - \frac{n^3}{3!}.\frac{(n+1)(n+2)(n+3)}{4^{n+4}} + \ldots\ldots \right\}$$

Solution by Martyn M. Thomas, M.A.

Let $x = e^{-y}$, so that $dx = -e^{-y}dy$, and limits for y are ∞ to 0.

$$\therefore \int_0^1 x^{x^x} dx = \int_0^\infty e^{-ye^{-ye^{-y}}} \times e^{-} dy$$

$$= \int_0^\infty \sum_{n=0} (-1)^n \left(\frac{y^n e^{-nye^{-y}}}{n!} \right) e^{-y} dy$$

$$= \int_0^\infty \sum_{n=0}^\infty (-1)^n . \frac{y^n}{n!} \left\{ 1 - \frac{nye^{-y}}{1!} + \frac{n^2y^2e^{-2y}}{2!} - \frac{n^3y^3e^{-3y}}{3!} + \ldots \right\} e^{-y} dy$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^\infty \left(e^{-y} y^n - \frac{n}{1!} e^{-2y} y^{n+1} + \frac{n^2}{2!} e^{-3y} y^{n+2} + \ldots \right) dy$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left[\frac{n!}{1^{n+1}} - \frac{n}{1!} \frac{(n+1)!}{2^{n+2}} + \frac{n^2}{2!} . \frac{(n+2)!}{3^{n+3}} - \ldots \right]$$

$$= \sum_{n=0}^\infty (-1)^n \left[\frac{1}{1^{n+1}} - \frac{n}{1!} . \frac{(n+1)}{2^{n+2}} + \frac{n^2}{2!} . \frac{(n+1)(n+2)}{3^{n+3}} + \ldots \right];$$

Question 1034.

(V. Ramaswami Aiyar, M.A.) :—Prove that the locus of the double foci of a system of confocal bicircular quartics consists of the two circular cubics which form the extreme members of the system.

Solution by M. Bhimasena Rao.

Let A, B, C, H, be the centres of inversion; DEF the ortho-centric triangle of ABCH, and P, Q, R, S the foci situated on a circle of inversion. The circular cubics of the system have for their focal conics the two parabolas through P, Q, R, S. The axes of the parabolas are at right angles, for P, Q, R, S are concyclic. If C_1 and C_2 be the centres of the

parabolas, the circular cubic the centre of whose focal parabola is C_1 passes through C_2 and *vice versa*. Both the cubics pass through the seven points A, B, C, H, D, E, F (Basset: *Cubic and Quartic Curves*). Since A, B, C, H are the in- and the ex-centres of the triangle DEF, the cubics are the loci of isogonal conjugate points, with respect to DEF, on parallel rays through C_1 and C_2.

The double foci of a bicircular quartic being the foci of its focal conic, we have to find the locus of the foci of the pencil of conics PQRS. Since P, Q, R, S are concyclic, the axes of the conics are fixed in direction and hence pass through C_1 and C_2—the centres of the parabolas of the pencil. If H is the centre of the circle PQRS, ABC is the diagonal triangle of PQRS and the focal conics are self-conjugate with respect to ABC. The foci of each conic are therefore isogonal conjugate points with respect to DEF, the orthocentrictriangle of ABC. (Question 13, Progress Report, Indian Mathematical Club), and being situated on lines passing through fixed points C_1 and C_2 at infinity, lie on the two circular cubics ABCHDEFC_1 and ABCHDEFC_2. Hence the theorem.

Question 1046.

(S. Krishnasawmi Aiyangar):—At every point O of a plane curve, a parabola and a rectangular hyperbola of closest contact with it are drawn. If the common normal meets the conics at P and Q, then

$$9.QO^3 + 4\,PO\,\rho_1{}^2 = 0,$$

where ρ_1 is the radius of curvature at the corresponding point of the evolute to the curve.

Solution by Martyn M. Thomas, Kewalramani, Hemraj and V. Tiruvenkatachari.

Choosing the tangent and normal at O as axes of x and y, the eqn. to conics having four point contact with it at O will be

$$2y = ax^2 + 2hxy + by^2,$$

where a and h will be the same for all the conics ; for,

$$2\rho = Lt.\ \frac{x^2}{y} \text{ at } O = \frac{2}{a};$$

and the centres of all the conics lie on the line

$$ax + hy = 0,$$

which is inclined in front of the normal at O at an angle $\tan^{-1}\left(\frac{1}{3}\frac{d\rho}{ds}\right)$

so that $$-\frac{h}{a} = \frac{1}{3}\left(\frac{d\rho}{ds}\right).$$

$$\therefore\ h = -\frac{1}{3\rho}\left(\frac{d\rho}{ds}\right).$$

Now b is different for different conics ; for a parabola $h^2 - ab = 0$ and for a rect. hyperbola $a + b = 0$.

Where the common normal meets these two conics

$$OP = \frac{2}{b}, \text{ as obtained from the parabola}$$

$$= \frac{2a}{h^2} = 18\rho \Big/ \left(\frac{d\rho}{ds}\right)^2 = 18\rho^3 \Big/ \left(\frac{d\rho}{d\psi}\right)^2 = \frac{18\rho^3}{\rho_1^{\,2}};$$

and $\quad OQ = \frac{2}{b}$, as obtained from the rect. hyperbola

$$= -\frac{2}{a} = -2\rho.$$

Eliminating ρ, we have, $\qquad 9OQ^3 + 4OP\,\rho_1^{\,2} = 0.$

Question 1047.

(S. KRISHNASAWMI AIYANGAR) :—If ρ be the radius of curvature at any point of a cardioide and ρ_0, ρ_α, $\rho_{\alpha\beta}$ at the corresponding points of its evolute, α-evolute, and the β-evolute of the α-evolute, show that

$$\rho_{\alpha\beta} = \rho_\alpha \sin\beta + \rho_0 \sin\alpha\cos\beta - \frac{1}{9}\rho\cos\alpha\cos\beta.$$

Solution by Martyn M. Thomas.

It has been shown, in Question 854, that for a curve in general,

$$\rho_\alpha = \rho\left(\frac{d\rho}{ds}\cos\alpha + \sin\alpha\right)$$

$$\rho_{\alpha\beta} = \rho\left(\frac{d\rho}{ds}\cos\alpha + \sin\alpha\right)\left(\frac{d\rho}{ds}\cos\beta + \sin\beta\right) + \rho\frac{d^2\rho}{ds^2}\cos\alpha\cos\beta.$$

$$\therefore \quad \rho_{\alpha\beta} = \rho_\alpha\sin\beta + \rho\frac{d\rho}{ds}\sin\alpha\cos\beta + \rho\cos\alpha\cos\beta\left\{\left(\frac{d\rho}{ds}\right)^2 + \rho\frac{d^2\rho}{ds^2}\right\}.$$

In the case of a cardioide,

$$r = a(1 - \cos\theta),\ \rho = \frac{4}{3}a\sin\frac{\theta}{2},$$

$$\frac{d\rho}{ds} = \frac{1}{3}\cot\frac{\theta}{2},\ \rho\frac{d^2\rho}{ds^2} = -\frac{1}{9}\operatorname{cosec}^2\frac{\theta}{2}.$$

$$\therefore \quad \rho_{\alpha\beta} = \rho_\alpha\sin\beta + \frac{d\rho}{d\psi}\sin\alpha\cos\beta + \rho\cos\alpha\cos\beta\left(-\frac{1}{9}\right)$$

$$= \rho_\alpha\sin\beta + \rho_0\sin\alpha\cos\beta - \frac{1}{9}\rho\cos\alpha\cos\beta.$$

Question 1054.

(MARTYN M. THOMAS) :—If s_r denote the sum of the squares of the fractions $\frac{1}{1}, \frac{1}{2}, \dots \frac{1}{r}$, and p_r denote the sum of their products taken two at a time, show that

$$^{n}C_1 s_1 - {}_nC_2(s_2 + p_2) + {}_nC_3(s_3 + p_3) - \dots$$
$$+ (-1)^{r-1}\,{}_nC_r(s_r + p_r) = \frac{1}{n^2}.$$

Solution by M. V. Ramakrishnan.

[The question is incorrect. The summation should extend to n terms.]

$${}_nC_1 s_1 - {}_nC_2\,(s_2 + p_2) + \dots\dots + (-1)^{n-1}{}_nC_n(s_n + p_n)$$
$$= s_1 + (n-1)\,\{s_1 - (s_2 + p_2)\}$$
$$- \frac{(n-1)(n-2)}{2\,!}\,\{(s_2 + p_2) - (s_3 + p_2)\} + \dots\dots \text{to } n \text{ terms.}$$

But

$$(s_r + p_r) - (s_{r-1} + p_{r-1}) = \frac{1}{r^2} + \frac{1}{r} + \frac{1}{2\,.\,r} + \dots + \frac{1}{(r-1)r}.$$

Hence if S denote the sum of the series

$$S = s_1 - \frac{(n-1)}{2}\left(\tfrac{1}{1} + \tfrac{1}{2}\right) + \frac{(n-1)(n-2)}{3\,!}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}\right) - \dots\dots$$

$$= \frac{1}{n}\left\{ns_1 - \frac{n(n-1)}{2\,!}\left(\tfrac{1}{1} + \tfrac{1}{2}\right) + \frac{n(n-1)(n-2)}{3\,!}\left(\tfrac{1}{1} + \tfrac{1}{2} + \tfrac{1}{3}\right) - \dots\dots\right\}$$

$$= -\frac{1}{n}\left\{(-t_1) - (n-1)(t_1 - t_2) + \frac{(n-1)(n-2)}{2\,!}(t_2 - t_3) - \dots\right\},$$

where $t_r = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{r}$.

$$= \frac{1}{n^2}\left\{{}_nC_1 - {}_nC_2 + \dots + (-1)^{n-1}{}_nC_n\right\} = \frac{1}{n^2}.$$

Question 1062.

(C. KRISHNAMACHARI) :—Prove that

$$1 + \binom{n-1}{1}\tfrac{1}{3} + \binom{n-1}{2}\tfrac{1}{5} + \dots\dots + \frac{1}{2n-1}$$

$$= 2^{n-1}\left\{\frac{1}{2n-1} + \frac{n-1}{(2n-1)(2n-3)} + \dots + \frac{(n-1)!}{(2n-1)(2n-3)\dots 1}\right\}.$$

Solution by M. V. Ramakrishnan.

The left side of the equality is equal to the coefft. of x^{2n-1} in the expansion of $\frac{1}{2}(1+x^2)^{n-1}\log\frac{1+x}{1-x}$.

Hence if

$$A_0+A_1x+\ldots+A_{2n-1}x^{2n-1}+\ldots=\tfrac{1}{2}(1+x^2)^{n-1}\log\frac{1+x}{1-x}, \quad (1)$$

then $A_{2n-1}=1+\binom{n-1}{1}\frac{1}{3}+\binom{n-1}{2}\frac{1}{5}+\ldots+\frac{1}{2n-1}$.

Differentiating (1) and equating coefft. of x^{2n-2}, we get

$$(2n-1)A_{2n-1}=\text{coefft. of } x^{2n-2} \text{ in } \frac{(1+x^2)^{n-2}}{1-x^2}$$

$$+\text{ coefft. of } x^{2n-2} \text{ in } (n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x}$$

$$=2^{n-1}+\text{coefft. of } x^{2n-2} \text{ in } (n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x}.$$

$$\therefore (2n-1)A_{2n-1}-2^{n-1}=\text{coefft. of } x^{2n-2} \text{ in } (n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x}.$$

Hence, if $B_{2n-1}=(2n-1)A_{2n-1}-2^{n-1}$, we get

$$B_0+B_1x+\ldots+B_{2n-1}x^{2n-2}+\ldots=(n-1)x(1+x^2)^{n-2}\log\frac{1+x}{1-x},$$

or $$\frac{B_0}{x}+B_1+\ldots+B_{2n-1}x^{2n-3}+\ldots=(n-1)(1+x^2)^{n-2}\log\frac{1+x}{1-x}.$$

Differentiating and equating coefft. of x^{2n-3}, we get

$$(2n-3)B_{2n-1}=2^{n-1}+\text{coefft. of } x^{2n-3} \text{ in } 2(n-1)(n-2)x(1+x^2)^{n-3}\log\frac{1+x}{1-x}.$$

Repeating this process $(n-1)$ times, we get the result.

QUESTIONS FOR SOLUTION.

1113. [SELECTED]:—An ellipse of eccentricity $\sin 2\alpha$, passes through the focus of a parabola of latus rectum $4a$ and has its foci on the curve. Show that the major axis envelopes the parabola

$$y^2 = 4a(1 - \tan^2\alpha)(x - a\tan^2\alpha);$$

and that the minor axis envelopes the semicubical parabola

$$27a\sec^2\alpha\, y^2 = 4(x - a - a\sec^2\alpha)^3.$$

1114. (T. P. TRIVEDI, M.A. LL. B):—If p, q are any two integers selected at random; prove that the probability that the fraction (p/q) is in its lowest terms is $6/\pi^2$.

1115. (T. P. TRIVEDI, M.A., LL. B):—Prove that

$$\text{(i)} \quad \frac{2^2-1}{2^2+1}\cdot\frac{3^2-1}{3^2+1}\cdot\frac{5^2-1}{5^2+1}\cdot\frac{7^2-1}{7^2+1}\ldots ad\ inf. = \frac{2}{5},$$

$$\text{(ii)} \quad \frac{2^4-1}{2^4+1}\cdot\frac{3^4-1}{3^4+1}\cdot\frac{5^4-1}{5^4+1}\cdot\frac{7^4-1}{7^4+1}\ldots ad\ inf. = \frac{6}{7},$$

where 2, 3, 5, etc. are all the prime numbers.

1116. (Prof. K. J. SANJANA):—If r be a positive integer (including zero) such that $(x - r\pi)$ always lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, shew that

$$\text{(i)} \quad \int_0^\infty \frac{x - r\pi}{x}\,dx = \frac{\pi}{2}\log 2,$$

$$\text{and (ii)} \int_0^\infty (-1)^r \frac{x - r\pi}{x}\,dx = 2\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots ad\ inf.\right)$$

1117. (Prof. K. J. SANJANA):—If a given function of x, y, z be transformed by the substitutions

$$r = \tfrac{1}{2}\log(x^2 + y^2 + z^2),\ \phi = \tan^{-1}\frac{y}{x},\ \theta = \tan^{-1}\frac{(x^2+y^2)^{\frac{1}{2}}}{z},$$

prove that the operation $(x^2 + y^2 + z^2)\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}\right)$ is transformed into $\left(\frac{\delta^2}{\delta r^2} + \frac{\delta^2}{\delta \theta^2} + \operatorname{cosec}^2\theta\,\frac{\delta^2}{\delta\phi^2} + \frac{\delta}{\delta r} + \cot\theta\,\frac{\delta}{\delta\theta}\right)$.

1118. (N. DORAI RAJAN):—Show that in the cubical parabola $x^3 = a^2y$, all tangent chords are trisected by the y —axis.

1119. (N. Dorai Rajan):—A triangle having its angular points on a limacon has its orthocentre at the node. Show that its circumcircle touches a fixed circle.

1120. (N. Dorai Rajan):—Show that the curve whose (p, r) equation is $r = a + p$, is an involute of an involute of a circle, *i.e.*, a 2nd involute of a circle. Find its polar equation and trace the curve.

1121. (N. Dorai Rajan):—Constant lengths a are measured along the tangents to a curve. If the locus of the extremities be a circle of radius a, show that the curve is the inverse of an involute of a circle of radius $2a$, the radius of inversion being equal to $2a$.

1122. (V. Ramaswami Aiyar):—If the joins of four concyclic points A, B, C, D taken in opposite pairs intersect in P, Q, R, prove that the nine points circle of PQR passes through the centroid of the points A, B, C, D.

1123. (G. S. M.):—Find a formula for the number of regions into which space of three dimensions is divided by n planes.

1124. (G. S. M.):—Prove that

$$1 + (r, 1)\frac{n}{m} + (r, 2)\frac{n(n-1)}{m(m-1)} + \ldots\ldots \text{ to } (n+1) \text{ terms}$$

$$= \frac{(m+1)(m+2)\ldots\ldots(m+r)}{(m-n+1)(m-n+2)\ldots\ldots(m-n+r)},$$

where $(r, 1)$, $(r, 2)$, ... are the *figurate* numbers of the rth order.

1125. (Communicated by G. S. M.):—Prove that

$$n!\, D^{-(n+1)}(u) = x^n D^{-1}(u) - c_1 x^{n-1} D^{-1}(xu) + \ldots\ldots + (-1)^n D^{-1}(x^n u),$$

where $D \equiv \dfrac{d}{dx}$ and $u \equiv f(x)$.